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# Existence and Multiplicity of Solutions for a Coupled Nonlinear Schrodinger System (Nonlinear Analysis and Convex Analysis)

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# Existence and Multiplicity of Solutions for a Coupled Nonlinear Schrödinger System

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## 1. Introduction

In the present paper, we consider the multiple existence of non-radial positive solutions of coupled Schrödinger system

$$(P) \quad \begin{cases} -\Delta u + \mu_1 u &= u^3 + \beta uv^2 & \text{in } \mathbb{R}^3 \\ -\Delta v + \mu_2 v &= v^3 + \beta u^2 v & \text{in } \mathbb{R}^3 \end{cases}$$

where  $\mu_1, \mu_2 > 0$  and  $\beta \in \mathbb{R}$ .

Coupled nonlinear Schrödinger system (P) models many physical problems. In nonlinear optics, the phenomenon in Kerr-like photorefractive media is described by system (P) (cf. [2]). In this case, the solution  $u$  and  $v$  denote the components of the beam in Kerr-like photorefractive media, and the coupling constant  $\beta$  is the interaction between two components  $u$  and  $v$ . In case  $\beta > 0$ , the interaction is attractive, while the interaction is repulsive if  $\beta < 0$ . The bimodal pulse in optical fibers under birefringent effects is also governed by system (P) (cf. [16]). It is also known that system (P) is a model for a mixture of two Bose-Einstein condensates (cf. [7]).

Motivated by these physical interest, the existence of solutions of (P) has been investigated by several authors. In the case that  $\beta > 0$ , problem (P) was studied by Ambrosetti & Colorado [3], Maia, Montefusco & Pellacci [15] and Lin & Wei [12]. They proved the existence of least energy solutions  $(u, v)$  of (P) with  $u, v > 0$ . By the result of Troy (cf. [22]), we know that in this case, all positive solutions  $(u, v)$  of (P) satisfying

$$(1.1) \quad u(x) \longrightarrow 0, v(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty$$

are radially symmetric functions. On the other hand, in case that  $\beta < 0$ , it is known that there is no least energy solution of (P) (cf. [12]). Moreover, positive solutions of (P) satisfying (1.1) is not always radial. In case that  $\beta < 0$  and  $|\beta|$  is small, there are positive solutions with one component concentrating on the origin and the other component concentrating around a regular polygon (cf. [14]). The existence of non radial positive solutions was also considered in [24] for the case that  $\beta < 0$  and  $\mu_1 = \mu_2$ . In this case, there are infinitely many nonradial solutions if  $\beta < -1$ . Recently, Sirakov [21]

established the existence of ground state solutions of (P) in the case that coefficients of nonlinear terms  $u^3$  and  $v^3$  in (P) are different.

On the other hand, the existence of sign changing solutions of nonlinear scalar elliptic problem

$$(1.2) \quad -\Delta u + u = |u|^{p-1} u, \quad u \in H_0^1(\Omega)$$

has been investigated by many authors in the last decade. Here  $\Omega$  is a domain in  $R^N$  ( $N \geq 3$ ), and  $p \in (1, (N+2)/(N-2)]$ . We refer to [5], [6] and [17] for related results of sign changing solutions of (1.2). The existence of sign changing solutions of (P) with  $\beta > 0$  was considered in [10]. For the problem (P) with  $R^3$  replaced by a bounded domain, we refer to [13] and [19].

In the present paper, we first see the multiple existence of solutions of problem (P) in the case that  $\beta > 0$ . Next we consider the case that  $\beta < 0$  and  $|\beta|$  is small, i.e. the case that the interaction of two solutions are small and repulsive. We will show the multiple existence of nonradial solutions of (P) in this case with  $\mu_1 \neq \mu_2$ . Our results improve the results in [14]. (See Remark 2 and Remark 3).

To state our main results, we need some notations. We denote by  $B_r(x)$  the open ball in  $R^3$  centered at  $x \in R^3$  with radius  $r > 0$ . The inner product in  $R^3$  is denoted by  $\langle \cdot, \cdot \rangle_{R^3}$ . We put  $H = H^1(R^3)$  and  $H = H \times H$ . We set  $\mu_0 = 1$ . We denote by  $\|\cdot\|_{\mu_i}$  the norm of  $H$  defined by  $\|u\|_{\mu_i}^2 = \int_{R^3} (|\nabla u|^2 + \mu_i |u|^2) dx$  for  $u \in H$  and  $i \in \{0, 1, 2\}$ . For simplicity of notations, we put  $|u(x)|_{\mu_i}^2 = |\nabla u(x)|^2 + \mu_i |u(x)|^2$  for  $u \in H$  and  $x \in R^3$ . For each function  $u \in H$ , we set  $u^+(x) = \max\{u(x), 0\}$ ,  $u^-(x) = \max\{-u(x), 0\}$ . For each  $p \geq 1$ , we denote by  $|\cdot|_p$  the norm of the space  $L^p(R^3)$ . The Hilbert space  $H$  is equipped with the norm defined by  $\|U\|^2 = \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2$  for  $U = (u, v) \in H$ . We recall that for each  $i \in \{0, 1, 2\}$ , problem

$$(P_i) \quad \begin{cases} -\Delta u + \mu_i u &= u^3 & \text{in } R^3 \\ u(x) &> 0 & \text{in } R^3 \\ u(x) &\longrightarrow 0 & \text{as } |x| \longrightarrow \infty \end{cases}$$

has a radial solution, denoted by  $U_i$  (cf. [8], [11]). The function  $U_i$  is the unique smooth solution of  $(P_i)$  up to translation. Moreover we know that  $U_i$  satisfies  $I_i(U_i) = c_i = \min\{I(v) : v \in S_i\}$ , where  $I_i$  is the functional associated with problem  $(P_i)$  defined by

$$(1.3) \quad I_i(v) = \frac{1}{2} \|v\|_{\mu_i}^2 - \frac{1}{4} |v^+|_4^4 \quad \text{for } v \in H$$

and  $S_i$  is the set defined by

$$S_i = \left\{ v \in H : \|v\|_{\mu_i}^2 = |v^+|_4^4 \right\} \quad \text{for } i = 1, 2.$$

For each  $x \in \mathbb{R}^3$  and  $i \in \{0, 1, 2\}$ , we put  $U_{i,x}(\cdot) = U_i(\cdot - x)$ . It is also known that

$$(1.4) \quad |U_i(x)|_{\mu_i} |x| \exp(\sqrt{\mu_i} |x|) \rightarrow c > 0, \text{ as } |x| \rightarrow \infty \quad \text{for } i \in \{0, 1, 2\}.$$

(cf. [11]). For each  $u \in L^4(\mathbb{R}^3)$ , we put  $\hat{u}(x) = \int_{B_1(x)} |u(x)|^4 dx$  for  $x \in \mathbb{R}^3$ . Then from (1.4), we can choose  $R_0 > 0$  such that

$$(1.5) \quad \hat{U}_i(z) < \frac{|\hat{U}_i|_\infty}{3} \quad \text{for all } z \in \mathbb{R}^3 \setminus B_{R_0}(0) \text{ and } i \in \{1, 2\}.$$

Since we consider the case that  $\mu_1 \neq \mu_2$ , we may assume without any loss of generality that  $\mu_2 < \mu_1$ . We can now state our main results.

**THEOREM 1.** *Suppose that the following condition holds:*

$$0 < 2\sqrt{\mu_2} < \sqrt{\mu_1}.$$

*Then there exists  $\beta_0 > 0$  such that for each  $\beta \in (0, \beta_0)$ , problem (P) possesses at least one ground state solution  $U_0 \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$  and one nonradial sign changing solution  $U \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ .*

**THEOREM 2.** *Suppose that  $\sqrt{\mu_1/\mu_2}$  is irrational. Then for each  $i \in \{2, 4, 6, 8, 12, 20\}$ , there exists  $\beta_i \in (-1, 0)$  such that for each  $\beta \in (\beta_i, 0)$ , there exists a positive solution  $U_i \in H$  of (P) such that  $ic_1 + c_2 < \Phi(U_i) < ic_1 + 2c_2$  and  $U_i$  has the form*

$$(1.6) \quad U_i = (U, V) = \left( \sum_{j=1}^i U_{1,x_j} + u, U_2 + v \right)$$

*where  $\{x_1, x_2, \dots, x_i\}$  forms a regular  $i$ -polyhedra in  $\mathbb{R}^3$  in case  $i \neq 2$  and  $x_1 = -x_2$  in case that  $i = 2$ , and  $u, v \in H$  such that  $\|(u, v)\|$  is so small that*

$$(1.7) \quad \hat{U}(z) < \frac{1}{2} |\hat{U}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus (\cup_{j=1}^i B_{R_0}(x_j)) \text{ and}$$

$$\hat{V}(z) < \frac{1}{2} |\hat{V}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0).$$

**REMARK 1.** *The expression (1.6) of  $U_i = (U, V)$  is unique when  $\|(u, v)\|$  is so small that condition (1.7) holds, i.e., for each  $U_i, (x_1, x_2, \dots, x_i)$  is uniquely determined.*

**REMARK 2.** *The assertion of Theorem 1 implies that for  $\beta < 0$  with  $|\beta|$  sufficiently small, problem (P) possesses at least 6 nonradial positive solutions. In [14], the existence of positive solutions of (P) of the form (1.6) was established in the case that  $\{x_1, x_2, \dots, x_i\}$  forms regular cube or tetrahedra under the assumption*

$$\sqrt{\frac{\mu_1}{\mu_2}} < \begin{cases} \frac{\sqrt{3}}{3} & \text{for the cube} \\ \frac{\sqrt{3}}{2} & \text{for the tetrahedra.} \end{cases}$$

Our argument employed in this paper does not require the ratio of  $\sqrt{\mu_1}$  and  $\sqrt{\mu_2}$ .

**THEOREM 3.** *Suppose that  $\sqrt{\mu_1/\mu_2}$  is irrational. Then for each  $k \in \mathbb{N}$ , there exists  $\tilde{\beta}_k \in (-1, 0)$  such that for each  $\beta \in (\tilde{\beta}_k, 0)$ , the problem (P) has a positive solution  $\tilde{u}_k$  such that  $kc_1 + c_2 < \Phi(\tilde{u}_k) < kc_1 + 2c_2$  and  $\tilde{u}_k$  has the form*

$$(1.8) \quad \tilde{u}_k = (U, V) = \left( \sum_{j=1}^k U_{1,x_j} + u, U_2 + v \right)$$

where  $\{x_1, x_2, \dots, x_k\} \subset \mathbb{R}^3$  form a regular  $k$ -polygon in a two dimensional subspace of  $\mathbb{R}^3$ , and  $u, v \in H$  such that  $\|(u, v)\|$  is so small that

$$\hat{U}(z) < \frac{1}{2} \left| \hat{U} \right|_{\infty} \quad \text{for } z \in \mathbb{R}^3 \setminus (\cup_{j=1}^k B_{R_0}(x_j)) \quad \text{and} \quad \hat{V}(z) < \frac{1}{2} \left| \hat{V} \right|_{\infty} \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0).$$

**REMARK 3.** *The existence of positive solutions of (P) of the form (1.8) was proved in [14] in the case that the spacial dimension is 2 and  $\mu_1, \mu_2$  satisfy*

$$\sqrt{\frac{\mu_1}{\mu_2}} < \sin \frac{\pi}{k}.$$

We give a sketch of the proof of Theorem 2 for the case  $i = 2$ . The proofs of Theorem 2 for  $i \neq 2$  and the proof of Theorem 3 are slight modifications of that of the case  $i = 2$  of Theorem 2. The detail of the proofs can be found in [9].

## 2. Preliminaries

Throughout the rest of this paper, we assume that  $\sqrt{\mu_1/\mu_2}$  is irrational.

For each  $u, v \in H = H^1(\mathbb{R}^3)$ , we put  $\langle u, v \rangle = \int_{\mathbb{R}^3} uv$ . We denote by  $\langle \cdot, \cdot \rangle_{\mu_i}$  the inner product of  $H$  defined by  $\langle u, v \rangle_{\mu_i} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \mu_i uv)$  for  $u, v \in H$  and  $i \in \{0, 1, 2\}$ . The inner product of  $H$  is defined by  $\langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{\mathbb{H}} = \langle U_1, U_2 \rangle_{\mu_1} + \langle V_1, V_2 \rangle_{\mu_2}$  for  $U_1 = (U_1, V_1), U_2 = (U_2, V_2) \in H$ . For each  $u \in L^4(\mathbb{R}^3)$ , we put  $\Omega(u) = \left\{ x \in \mathbb{R}^3 : \hat{u}(x) \geq \frac{|\hat{u}|_{\infty}}{2} \right\}$  and

$$B(u) = \frac{\int_{\Omega(u)} x (\hat{u}(x) - \frac{|\hat{u}|_{\infty}}{2}) dx}{\int_{\Omega(u)} (\hat{u}(x) - \frac{|\hat{u}|_{\infty}}{2}) dx}.$$

The mapping  $B$  is called generalized barycenter, which is introduced in [18] (cf. also [4]). By Sobolev's embedding theorem ([1]), for  $p \in [2, 6]$  there exists  $m_p > 0$  such that

$$|z|_{L^p(B_r(0))} \leq m_p \left( |\nabla z|_{L^2(B_r(0))}^2 + \mu_i |z|_{L^2(B_r(0))}^2 \right)^{1/2}$$

for  $r > 1, z \in H^1(B_r(0))$ , and  $i \in \{0, 1, 2\}$ . For each  $a \in R$ , and a functional  $F : H \rightarrow R$ , we denote by  $F^a$  the level set defined by  $F^a = \{v \in \mathbb{H} : F^a(v) \leq a\}$ . The same notation is used for functionals defined on  $H$ .

It is easy to see that for each  $u \in H \setminus \{0\}$  with  $u^+ \neq 0$ , there exists a unique positive number  $t$  such that  $tu \in S_i$  (cf. [25]). It follows from the definitions of  $U_i$  that  $U_i(x) = \sqrt{\mu_i}U_0(\sqrt{\mu_i}x)$  on  $R^3$ . Then one can see that

$$(2.1) \quad c_1 = \sqrt{\mu_1}c_0 > c_2 = \sqrt{\mu_2}c_0.$$

Let  $i \in \{0, 1, 2\}$ . It is known that  $\{U_{i,x} : x \in \mathbb{R}^3\}$  is a nondegenerate critical set of  $I_i$  (cf. [23]). More precisely, we have there exists  $\lambda > 0$  such that

$$(2.2) \quad \|u\|_{\mu_i}^2 - 3 \langle U_i^2 u, u \rangle \geq \lambda \|u\|_{\mu_i}^2 \quad \text{for all } u \in \left\{ U_i, \frac{\partial U_i}{\partial x_1}, \frac{\partial U_i}{\partial x_2}, \frac{\partial U_i}{\partial x_3} \right\}^\perp.$$

We define a functional  $\Phi : H \rightarrow R$  associated with problem (P) by

$$\begin{aligned} \Phi(\mathcal{U}) &= \frac{1}{2}(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2) - \frac{1}{4}(|U^+|_4^4 + |V^+|_4^4) - \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ &= \Phi_1(\mathcal{U}) + \Phi_2(\mathcal{U}) \quad \text{for } \mathcal{U} = (U, V) \in \mathbb{H}, \end{aligned}$$

where

$$\Phi_1(\mathcal{U}) = \frac{1}{2} \|U\|_{\mu_1}^2 - \frac{1}{4} |U^+|_4^4 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2$$

and

$$\Phi_2(\mathcal{U}) = \frac{1}{2} \|V\|_{\mu_2}^2 - \frac{1}{4} |V^+|_4^4 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2.$$

Then a direct computation shows

$$\begin{aligned} \langle \nabla \Phi(\mathcal{U}), \nu \rangle_{\mathbb{H}} &= \left\langle \begin{pmatrix} \nabla_u \Phi(\mathcal{U}) \\ \nabla_v \Phi(\mathcal{U}) \end{pmatrix}, \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle_{\mathbb{H}} \\ &= \langle -\Delta U + \mu_1 U - (U^+)^3 - \beta U^+ (V^+)^2, W \rangle \\ &\quad + \langle -\Delta V + \mu_2 V - (V^+)^3 - \beta (U^+)^2 V^+, Z \rangle \end{aligned}$$

for  $U = (U, V), V = (W, Z) \in H$ . We put

$$\begin{aligned} \mathcal{M}_+ &= \{(U, V) \in \mathbb{H} \setminus \{0\} : \|U\|_{\mu_1}^2 = |U^+|_4^4 + \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2, \\ &\quad \|V\|_{\mu_2}^2 = |V^+|_4^4 + \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2\}. \end{aligned}$$

Then one can see that  $U = (U, V) \in \mathcal{M}_+$  is a critical point of  $\Phi$  if and only if  $U$  is a positive solution of problem (P). From the definition, we have

$$(2.3) \quad \Phi_1(\mathcal{U}) = \frac{1}{4} \|U\|_{\mu_1}^2, \Phi_2(\mathcal{U}) = \frac{1}{4} \|V\|_{\mu_2}^2 \quad \text{and} \quad \Phi(\mathcal{U}) = \frac{1}{4} (\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2)$$

for  $U = (U, V) \in \mathcal{M}_+$ . We also have that for each  $U = (U, V) \in \mathcal{M}_+$ , there exists  $(s, t) \in R^+ \times R^+$  such that  $(sU, tV) \in \mathcal{M}_+$ . In fact, for each  $U = (U, V), U \neq$

$0, V \neq 0, (sU, tV) \in M_+$  if and only if

$$\begin{pmatrix} s^2 \\ t^2 \end{pmatrix} = A^{-1} \begin{pmatrix} \|U\|_{\mu_1}^2 \\ \|V\|_{\mu_2}^2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} |U^+|_4^4 & \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 & |V^+|_4^4 \end{pmatrix}.$$

Since  $\beta \in (-1, 0)$ , we have by the Schwartz's inequality that  $A^{-1}$  exists and then there exists a unique solution  $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$ . For given  $U = (U, V) \in H$  with  $U \neq 0, V \neq 0$ , we put  $NU = N(U, V) = (N_1U, N_2V) = (sU, tV) \in M_+$ .

Now to prove Theorem 1 for the case that  $i = 2$ , we define  $H_2 \subset H, H_2 \subset H$  and  $M_2 \subset M_+$  by

$$H_2 = \{u \in H : u(x) = u(-x) \text{ for } x \in \mathbb{R}^3\}, \quad \mathbb{H}_2 = H_2 \times H_2,$$

and

$$\mathcal{M}_2 = \mathcal{M}_+ \cap \mathbb{H}_2.$$

Since  $\sqrt{\mu_1/\mu_2}$  is irrational, we can choose  $\delta_2 \in (0, c_2)$  so small that

$$(2.4) \quad c_1 + kc_2 \notin [2c_1 + c_2 - \delta_2, 2c_1 + c_2 + \delta_2] \quad \text{for all } k \in \mathbb{N}.$$

The following Lemmata are crucial for our argument.

LEMMA 1. (1) *There exists  $\beta_1 \in (-1, 0)$  such that for each  $\beta \in (\beta_1, 0)$  and each critical point  $U \in M_2 \cap \Phi^{2c_1+2c_2}$  of  $\Phi$ ,*

$$\Phi(U) \in \cup_{i \geq 1, j \geq 1} [ic_1 + jc_2 - \delta_2/2, ic_1 + jc_2 + \delta_2/2].$$

(2) *Let  $\beta \in (\beta_1, 0)$  and  $\{U_n\} \subset M_2$  such that  $\lim_{n \rightarrow \infty} \nabla \Phi(U_n) = 0$  and  $\lim_{n \rightarrow \infty} \Phi(U_n) = 2c_1 + c_2 + \varepsilon$  with  $\varepsilon \in (0, \delta_2/2)$ . Then there exists a convergence subsequence  $\{U_{n_i}\} \subset \{U_n\}$ .*

LEMMA 2. *For given  $\varepsilon > 0$ , there exists  $\beta_\varepsilon \in (\beta_1, 0)$  such that for each  $\beta \in (\beta_\varepsilon, 0)$  and for each  $x \in \mathbb{R}^3 \setminus \{0\}$ ,*

$$(2.5) \quad \Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \varepsilon$$

a

### 3. Sketch of the Proof of Theorem 2 for $i = 2$ .

Throughout this section we assume that  $\beta \in (\beta_1, 0)$ . We put

$$b_R(U) = \int_{\mathbb{R}^3 \setminus B_R(0)} |U|_{\mu_1}^2 \quad \text{for } U \in H \text{ and } R > 0$$

and

$$\Lambda_{2,\varepsilon}(R) = \left\{ \mathcal{U} = (U, V) \in \Phi^{2c_1+c_2+\varepsilon} \cap \mathcal{M}_2 : b_R(U) \geq 8c_1 - \min \left\{ \frac{1}{2m_4^4}, c_1 \right\} \right\}$$

for each  $\varepsilon > 0$  and  $R > 0$ .

**PROPOSITION 1.** *For  $\varepsilon > 0$  sufficiently small, there exists  $(R_\varepsilon, \delta_\varepsilon, \alpha_\varepsilon, \gamma_\varepsilon) \in (\mathbb{R}^+)^4$  such that  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon = 0$  and each  $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$  has the form*

$$(3.1) \quad \mathcal{U} = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v)$$

where  $\alpha \in (1 - \alpha_\varepsilon, 1 + \alpha_\varepsilon)$ ,  $\gamma \in (1 - \gamma_\varepsilon, 1 + \gamma_\varepsilon)$ ,

$$(3.2) \quad |x| \geq R_\varepsilon, \quad x = \mathcal{B}(U|_{B_{R_0}(x)}), \quad \widehat{U}(z) < \frac{1}{2} \left| \widehat{U} \right|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus \bigcup_{i=\pm 1} B_{R_0}(ix),$$

$$(3.3) \quad \widehat{V}(z) < \frac{1}{2} \left| \widehat{V} \right|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0),$$

and

$$(3.4) \quad (u, v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_2\}^\perp \quad \text{with } \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2 \leq \delta_\varepsilon.$$

**REMARK 4.** *By (3.2) and the definition of  $\mathcal{B}$ , one can see that for each  $\mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon)$ ,  $(x, -x) \in \mathbb{R}^3 \times \mathbb{R}^3$  in (3.1) is uniquely determined, and the mapping  $\mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon) \rightarrow (x, -x) \in \mathbb{R}^3 \times \mathbb{R}^3$  is continuous. We define a continuous mapping  $\eta : \Lambda_{2,\varepsilon}(R_\varepsilon) \rightarrow \mathbb{R}^+$  by*

$$(3.5) \quad \eta(\mathcal{U}) = |x| \quad \text{for } \mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon).$$

We also need the following Proposition.

**PROPOSITION 2.** *There exists  $M_0 > 0$  satisfying that for  $\varepsilon > 0$  sufficiently small,*

$$\Phi(\mathcal{U}) \geq 2c_1 + c_2 - \beta M_0 e^{-2\sqrt{\mu_2}|x|} \quad \text{for each } \mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon),$$

where  $x \in \mathbb{R}^3$  such that  $\mathcal{U}$  has the form (3.1).



Now for  $x \in R^3 \setminus \{0\}$ , we define a class  $\Gamma_2(x) \subset C([0, 1], M_2)$  by

$$\Gamma_2(x) = \{p \in C([0, 1], M_2) : p(0) = \mathcal{N}(U_1, U_2), p(1) = \mathcal{N}(U_{1,x} + U_{1,-x}, U_2)\}$$

and put

$$c_2(x) = \inf_{p \in \Gamma_2(x)} \sup_{t \in [0, 1]} \Phi(p(t)).$$

We also note that from the definitions of  $\mathcal{N}$  and  $\Phi$ , we have that  $\mathcal{N}(U_{1,x} + U_{1,-x}, U_2) - (U_{1,x} + U_{1,-x}, U_2) \rightarrow 0$  in  $H$  as  $|x| \rightarrow \infty$  and then

$$(3.6) \quad \lim_{|x| \rightarrow \infty} \Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) = 2I_1(U_1) + I_2(U_2) = 2c_1 + c_2.$$

Based on the preliminary results above, we can prove Theorem 2 for  $i = 2$ .

**PROOF OF THEOREM 2.** Let  $\varepsilon \in (0, \delta_2/2)$  sufficiently small. Let  $\beta \in (\beta_\varepsilon, 0)$ . To complete the proof, it is sufficient to show that there exists  $\delta > 0$  and  $R > 0$  such that

$$(3.7) \quad 2c_1 + c_2 + \delta < c_2(x) < 2c_1 + c_2 + \delta_2/2 \quad \text{for } |x| > R.$$

In fact, if the inequalities above hold, we have by (3.6) that we can choose  $x \in R^3$  such that  $|x| > R$  and

$$\Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \delta.$$

That is  $\Phi(p(1)) < c_2(x)$  for all  $p \in \Gamma_2(x)$ . We also have  $\Phi(p(0)) < \frac{7}{4}c_1 + c_2$ . Then since the Palais-Smale condition holds by (2) of Lemma 1 on  $\Phi^{(2c_1+c_2, 2c_1+c_2+\delta_2/2)}$ , we have by a standard mountain pass argument that there exists a critical point  $U$  of  $\Phi$  with  $\Phi(U) = c_2(x)$ .

From the definition of  $\varepsilon$  and Lemma 2, one can see the pass  $p \in \Gamma_2(x)$  defined by

$$p(s) = \mathcal{N}(U_{1,sx} + U_{1,-sx}, U_2), \quad s \in [0, 1]$$

satisfies  $\max_{s \in [0, 1]} \Phi(p(s)) \leq 2c_1 + c_2 + \varepsilon$ . Then the second inequality of (3.7) holds. We now show that the first inequality of (3.7) holds. We first see that there exists  $\bar{R} > 2R_\varepsilon$  such that

$$(3.8) \quad b_{R_\varepsilon}(U) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\} \quad \text{for } \mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon) \text{ with } \eta(U) \geq \bar{R},$$

where  $\eta$  is the function defined by (3.5). By Proposition 1, each  $U = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$  has the form

$$(3.9) \quad \mathcal{U} = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v)$$

with  $\alpha \in (1 - \alpha_\varepsilon, 1 + \alpha_\varepsilon)$ ,  $\gamma \in (1 - \gamma_\varepsilon, 1 + \gamma_\varepsilon)$  and  $(u, v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_2\}^\perp$  with  $\|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2 \leq \delta_\varepsilon$ . Since  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = 0$ , we may assume that  $\varepsilon > 0$  is sufficiently small that

$$(3.10) \quad 8\alpha_\varepsilon^2 c_1 - \delta_\varepsilon > 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}.$$

Then noting that

$$b_{R_\varepsilon}(U) \geq \alpha^2 \|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 - \|u\|_{\mu_1}^2 - 2 \int_{B_{R_\varepsilon}(0)} |U_{1,x} + U_{1,-x}|_{\mu_1}^2$$

and

$$\|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 \longrightarrow 8c_1 \text{ and } \int_{B_{R_\varepsilon}(0)} |U_{1,x} + U_{1,-x}|_{\mu_1}^2 \longrightarrow 0, \text{ as } |x| \longrightarrow \infty,$$

we find by (3.10) that there exists  $\bar{R}$  such that for each  $U = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$  with  $\eta(U) \geq \bar{R}$ , (3.8) holds. Now we choose  $x \in \mathbb{R}^3$  so large that  $|x| > \bar{R}$ . Then

$$b_{R_\varepsilon}(\mathcal{N}_1(U_{1,x} + U_{1,-x})) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}.$$

Let  $p = (p_1, p_2) \in \Gamma_2(x)$  such that  $\sup_{t \in [0,1]} \Phi(p(t)) \leq 2c_1 + c_2 + \varepsilon$ . From the definition,

$$\eta(p_1(1)) = \eta(\mathcal{N}_1(U_{1,x} + U_{1,-x})) > \bar{R} \text{ and } b_{R_\varepsilon}(p_1(1)) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}.$$

On the other hand, recalling that  $\Phi_2(U) \geq c_2$ , we have that  $\Phi_1(U) \leq \frac{7}{4}c_1$ . Then by the definition of  $\varepsilon$ ,  $b_{R_\varepsilon}(p_1(0)) < 7c_1 \leq 8c_1 - \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}$ , there exists  $t \in (0, 1)$  such that  $b_{R_\varepsilon}(p_1(t)) = 8c_1 - \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}$ . Then by (3.8),  $\eta(p_1(t)) < \bar{R}$ . Therefore by the continuity of  $\eta$ , we have that there exists  $t_0 \in (0, t)$  such that  $\eta(p_1(t_0)) = \bar{R}$ . By Proposition 2, we have

$$\Phi(p(t_0)) \geq 2c_1 + c_2 + \beta M_0 e^{-2\sqrt{\mu_2}\bar{R}}.$$

Therefore we obtain that  $\sup_{t \in [0,1]} \Phi(p(t)) > 2c_1 + c_2 + \beta M_0 e^{-2\sqrt{\mu_2}\bar{R}}$ . Thus by the mountain pass theorem (cf. [20]), we find that there exists a critical point  $U$  of  $\Phi$  with  $\Phi(U) = c_2(x)$ .  $\square$

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